A Study of Causal Curves and Singularities in Lorentz Manifold with Path Topology

Synopsis

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1 Introduction

Various physical theories exhibit mathematical structure explaining physical contents of the theory. Euclidean space wherein Euclidean metric encodes the Euclidean geometry is an appropriate model to study classical mechanics, one of the oldest physical theories. Other physical theories viz., special theory of relativity, general theory of relativity and quantum theory were developed to explain several physical phenomena which had no explanation in classical Physics.

Spacetime, a continuum of space and time, is assumed to carry a causal structure to explain cause-effect relationship among its elements. It has been studied as a causal space with a causal boundary attached to it which represents both infinity and the part of the edge of spacetime at a finite distance, i.e., the singular points [41]. One of the admissibility conditions for the mathematical structure of spacetime is that no idealized observer can travel back into its own past [41]. Four-dimensional Minkowski space was considered as a mathematical framework for spacetime in special theory of relativity wherein null cones determined its causal structure. A physically significant topology defined for the Minkowski space is considered to be the one whose homeomorphism group coincides with the group generated by translations, Dilatations and linear operators which are consistent with the causal structure of Minkowski space. Since homeomorphism group of the Euclidean topology does not coincide with the Lorentz group, therefore it is not physically relevant. Zeeman proposed non-Euclidean topologies, viz., fine topology, space topology, time topology, *t* and the *s* topologies on the four-dimensional Minkowski space in order to make group of homeomorphisms physically significant [4, 38, 39]. These topologies were defined for the n-dimensional Minkowski space analogously and the topological study of the n-dimensional Minkowski space with respect to non-Euclidean topologies has been carried out [8, 9, 10, 11].

General theory of relativity generalizes special theory of relativity and treats gravity not as a force field but a built-in feature of spacetime. Gravity is invariably experienced by all objects in spacetime carrying masses and is the reason for spacetime having the curved structure instead of at. A smooth, non-orientable four-dimensional Lorentz manifold has been considered as a mathematical structure for spacetime in order to explain Einstein's general theory of relativity. Quantities associated with the Lorentz metric, for example the connection, the Riemann tensor, the Ricci tensor and others, have

physical interpretations and can be used to develop physics in this setting, and make testable and tested predictions, which to date have proven consistent with the available experimental data [34]. Lorentz manifold as a structure of spacetime on length scale is from 10^{-13} cm, the radius of an elementary, particle up to 10^{28} cm, the radius of the universe [41].

Homeomorphism group of the manifold topology defined on a Lorentz manifold is huge and has no physical significance. Path topology, introduced by Hawking on a four-dimensional Lorentz manifold, is a non-manifold topology which induces manifold topology on each timelike curve and whose homeomorphism group is consistent with the causal and conformal structure of the Lorentz manifold [42]. Various non-manifold topologies on a four-dimensional Lorentz manifold were introduced including physically relevant generalizations of the fine and time topologies which are referred to as Zeeman and geodesic topologies respectively [11, 23, 24, 27]. Path topology on the four-dimensional Minkowski space is same as the t-topology [18]. Manifold topology is strictly coarser than the Zeeman, geodesic and path topologies.

Singularity is defined as the region of spacetime where no known physical laws are applicable [41]. Penrose, Hawking, GFR Ellis, R. Geroch are some of the few names who have tremendously contributed towards the study of spacetime singularities [41, 25, 26, 29]. Mathematically, the occurrence of singularities was defined in terms of the existence of incomplete timelike and null geodesics in spacetime [41]. The concept of b-incompleteness as a generalization of g-incompleteness was introduced as a definition of singularity in spacetime [41]. A spacetime singularity is also given by imprisoned incompleteness which occurs only if strong causality is violated [41].

Singularity theorems given by Penrose and Hawking are concerned with deriving sufficient conditions for spacetime equipped with the manifold topology to be timelike and null geodesically incomplete [41]. Assuming suitable curvature conditions satisfied by the timelike geodesically complete spacetime endowed with the manifold topology, a topological characterization in terms of Lorentzian analogue of splitting theorem of Riemannian manifolds has been obtained [14, 15]. Global hyperbolicity and limit curve theorem are some of the essential ingredients of singularity theorems. Global hyperbolicity, the strongest causality condition has been studied independently and it signifies the absence of a particularly weak kind of naked singularity in spacetime [31, 32].

The mathematical analysis of spacetime singularities broadly constitutes the study of the behaviour of families of causal curves and the nature of various causal relations in the Lorentz manifold which are usually known as causality conditions. These studies have enabled us towards developing an understanding of the beginning and final fate of the universe.

The present proposal aims at the study pertaining to singularities of spacetime in the setting of Lorentz manifold with respect to a physically significant non-manifold topology, viz. the path topology.

2 Notation and Preliminaries

This section comprises notation, basic definitions and known results related to the present proposal. The symbols N and R denote the set of natural numbers and the set of real numbers respectively. For details, we refer to [34, 41, 45, 36, 21, 20, 42, 5, 15, 16].

For $x \in R^n$ and $\epsilon > 0$; $N_\epsilon^E(x)$ denotes the Euclidean open ball about x of radius ϵ given by the set $\{y \in R^n : d_E(x,y) < \epsilon\}$, where $d_E(x,y)$ is the Euclidean distance between x and y. For $n \in N$ and $n > 1$, the n-dimensional real vector space R^n with the bilinear form $g: R^n \times R^n \to R$; satisfying the following properties : (i) for all $x, y \in R^n$; $g(x,y) = g(y,x)$; i.e., the bilinear form is symmetric, (ii) if for all $y \in R^n$; $g(x,y) = 0$; then $x = 0$, i.e., the bilinear form is non-degenerate, and (iii) there exists a basis $\{e_0, e_1, \dots e_{n-1}\}$ for R^n with

$$g(e_i, e_j) \equiv \eta_{ij} = \begin{cases} 1 & i = j = 0 \\ -1 & i = j = 1,2 \dots n-1 \\ 0 & otherwise \end{cases}$$

is called the n-dimensional *Minkowski space* which will be denoted by L throughout. The bilinear form g is called the *Lorentz metric*. Elements of Minkowski space are called *events*. For an event, $x \equiv \sum_{i=0}^{n-1} x_i e_i$, the coordinate x_0 is called *time component* and the coordinates $x_1, x_2, \dots x_{n-1}$ are called *spatial components* of x relative to the basis $\{e_0, e_1, \dots e_{n-1}\}$. In terms of components, the Lorentz metric $g(x,y)$ of two events $x \equiv \sum_{i=0}^{n-1} x_i e_i$ and $y \equiv \sum_{i=0}^{n-1} y_i e_i$ is given by $x_0 y_0 - \sum_{i=1}^{n-1} x_i y_i$. Lorentz metric induces an indefinite characteristic quadratic form Q on L defined by $Q(x) = g(x,x)$. Thus $Q(x) = x_0^2 - \sum_{i=1}^{n-1} x_i^2$. The group generated by all linear operators on L which leave the quadratic form invariant is called the *Lorentz group*.

A Lorentz manifold is a pair (M, g), where M is an n-dimensional smooth manifold and g is a Lorentz metric, i.e., g associates to each point $p \in M$ a Lorentz metric on the n-dimensional tangent space $T_p(M)$.

2.1 Relevant notions in Minkowski space

An event $x \in L$ is called *timelike*, *lightlike* (also called *null*), or *spacelike*, accordingly as $Q(x)$ is positive, zero or negative. The sets $C^T(x) = \{y \in L: y = x \text{ or } Q(y-x) > 0\}$, $C^L(x) = \{y \in L: Q(y-x) > 0\}$ and $C^S(x) = \{y \in L: y = x \text{ or } Q(y-x) < 0\}$ are called the *time cone*, *light cone* (or *null cone*), and *space cone* at *x*. A straight line through *x* is called *timelike* or *light ray* or *spacelike*, according to which cone at *x* it lies in. For further details, we refer to [4]. The Euclidean topology on L is generated by the basis $B = \{N_\epsilon^E(x): x \in L, \epsilon > 0\}$. The collection $N_\epsilon^t(x) = N_\epsilon^E(x) \cap C^T(x)$, forms a local base for the family of neighbourhoods of $x \in L$. The topology generated by these neighbourhood systems is called the *t-topology*: $N_\epsilon^t(x)$ will be called the *t*-neighbourhood of radius ϵ. Hence $U \subseteq L$ is open with respect to the *t*-topology if and only if for every $x \in U$, there exists some $\epsilon > 0$ such that $N_\epsilon^t(x) \subseteq U$. It thus follows that $N_\epsilon^t(x)$ is open in L with respect to the *t*-topology, however it is not open with respect to the Euclidean topology. In other words, *t*-topology is strictly finer than the Euclidean topology on L. The collection $\{N_\epsilon^S(x): \epsilon > 0\}$, where $N_\epsilon^S(x) = N_\epsilon^E(x) \cap C^S(x)$, forms a basis for a topology on L known as *s-topology*. It is strictly finer than the Euclidean topology.

The *time topology* on L is defined to be the finest topology which induces the Euclidean topology on every time axis. Similarly, the *space topology* on L is defined to be the finest topology which induces the Euclidean topology on every space-axis. The time and space topologies are strictly finer than the *t* and *s* topologies, respectively. The topology on L which induces the Euclidean topology on every time axis and space axis is called *fine topology*. It is evident that the fine topology is coarser than the time and space topologies.

2.2 Relevant notions in Lorentz manifold

Spacetime, denoted by M, is considered to be a connected, Hausdorff, paracompact, C^∞ four-dimensional Lorentz manifold. A vector X in $T_p(M)$ is said to be timelike, causal or null according as $g(X,X) > 0, g(X,X) \geq 0$ or $g(X,X) = 0$.

There is an open neighbourhood $\widehat{N}$ of the origin of $T_p(M)$ such that $U \equiv exp(\widehat{N})$ is an open convex neighbourhood of p and exp restricted to $\widehat{N}$, is a diffeomorphism, where exp is the exponential mapping from $T_p(M)$ to M: U is called a normal neighbourhood of p. Further p possesses a neighbourhood basis of open convex normal neighbourhoods. Let the pre-image of $q \in U$ in $\widehat{N}$ under the exponential map be denoted by $(x^0(q), x^1(q), x^2(q), x^3(q))$. Every pair of points in U can be joined by a unique geodesic curve and the geodesics in U through p are the images of straight lines through the origin in $\widehat{N} \subseteq T_p(M)$. Define $B_U(p,\epsilon) = exp(B)$ where B is an open neighbourhood of the origin of $T_p(M)$ of radius $\epsilon > 0$. Then $x^i(p) = 0$ and $I^+(p,U) = \{q \in U: \left(x^0(q)\right)^2 - \left(x^1(q)\right)^2 - \left(x^2(q)\right)^2 - \left(x^3(q)\right)^2 > 0,\ x^0(q) > 0\}$, $I^-(p,U) = \{q \in U: \left(x^0(q)\right)^2 - \left(x^1(q)\right)^2 - \left(x^2(q)\right)^2 - \left(x^3(q)\right)^2 > 0,\ x^0(q) < 0\}$. Define $K(p,U) = I^+(p,U) \cup I^-(p,U) \cup \{p\}$.

The collection of subsets of M of the form $L_U(p,\epsilon) = B_U(p,\epsilon) \cup K(p,U)$ forms a basis for a topology known as path topology denoted by ρ. The group of homeomorphisms from M to M, with respect to ρ coincides with the group G of conformal diffeomorphisms of M and every such homeomorphism maps null geodesics onto null geodesics. Another significant topology *Alexandroff* topology $\mathcal{A}$ on M is generated by the collection of sets of the form $I^+(p,M) \cap I^-(q,M)$. *Geodesic* topology on M is defined as a generalization of the time topology on Minkowski space.

A curve in M is a continuous map from a connected set in R to M equipped with the manifold topology. A differentiable curve γ is said to be a timelike (or causal) if $g(\gamma'(t), \gamma'(t)) >$ (or ≥ 0), where $\gamma'(t) \in T_{\gamma(t)}(M)$.

A causality neighbourhood D of a point $p \in M$ is an open neighbourhood of p with respect to the manifold topology such that, whenever $\gamma: F \to M$ is a smooth causal path, $\gamma^{-1}(D)$ is connected, F is a

connected interval of R. M is strongly causal at p, if there exists a local neighbourhood basis at p consisting of causality neighbourhoods.

2.2.1 Limit Curve Theorem

Let $(\lambda_n)_{n\in N}$ be a sequence of causal curves. A point $p \in M$ is said to be a *convergence point* of $(\lambda_n)_{n\in N}$ if, given any open neighbourhood U of p, there exists an $n_0 \in N$ such that Image$(\lambda_n) \cap U \neq \emptyset$ for all $n > n_0$. A curve λ is said to be a *convergence curve* of $(\lambda_n)_{n\in N}$ if each $p \in$ Image(λ) is a convergence point. Similarly, p is said to be a *limit point* of $(\lambda_n)_{n\in N}$ if every open neighbourhood of p intersects infinitely many λ_n's. A curve λ is said to be a *limit curve* of $(\lambda_n)_{n\in N}$ if there exists a subsequence $(\lambda_{n_k})_{k\in N}$ for which λ is a convergence curve.

Theorem (*Hawking and Ellis*) Let $(\lambda_n)_{n\in N}$ be an infinite sequence of continuous causal curves which are (past, future) inextendible. If *x* is a limit point of $(\lambda_n)_{n\in N}$, then through *x* there is continuous causal curve which is (resp. past, future) inextendible and is a limit curve of the $(\lambda_n)_{n\in N}$ [5].

Beem et al. formulated a different definition of limit curve theorem which replaces the word "limit curve" with "convergence curve", defined as above. Minguzzi generalized the limit curve theorem for the case when curves are not inextendible and when there exists more than one limit point [5].

2.2.2 Study related to causal structure

A subset $S \subset M$ is said to be *achronal* if there do not exist $p, q \in S$ such that $q \in I^+(p)$, i.e., $I^+(S) \cap S = \emptyset$. A closed achronal set with edge$(S) = \emptyset$ is a three-dimensional, embedded C^0 submanifold of M. *Future domain of dependence* of a closed achronal set S, denoted $D^+(S)$, is defined by $\{p \in M$: Every past inextendible curve through p intersects $S\}$. Similarly, $D^-(S)$ is defined. $D(S) = D^+(S) \cup D^-(S)$ is called *domain of dependence*. A closed achronal set, say Σ, such that $D(\Sigma) = M$, is called a *Cauchy surface*. Also, the edge$(\Sigma) = \emptyset$, therefore a Cauchy surface is a three-dimensional, embedded C^0 submanifold of M.

A Lorentz manifold M is called globally hyperbolic if the strong causality assumption holds on M and if for any two points $p, q \in M$, $J^+(p) \cap J^-(q)$ is compact. Global hyperbolicity is also defined in terms of existence of a Cauchy surface in M. These two definitions have been proved to be equivalent.

In a globally hyperbolic Lorentz manifold the set of all continuous causal curves from p to q is compact.

We present below some results concerning Lorentz manifold.

Theorem 1 Let M be a globally hyperbolic spacetime. Then M is stably causal. Furthermore, a global time function, say f, can be chosen such that each level surface is a Cauchy surface. Thus M can be foliated by Cauchy surfaces and the topology of M is $R \times \Sigma$, where Σ denotes any Cauchy surface [34].

Theorem 2 Let M be a non-compact spacetime having non-positive timelike sectional curvatures and containing a compact spacelike hypersurface. Then M is either timelike geodesically incomplete or splits isometrically into the product $R \times S$, where (S, h) is a compact Riemannian manifold [16].

Let M be a non-compact spacetime which satisfies the strong energy condition and contains a compact smooth spacelike hypersurface S of constant mean curvature. Then M is either timelike geodesically incomplete or splits as in Theorem 2 [16].

Theorem 3 Let M be a connected globally hyperbolic space-time which satisfies strong energy condition for all timelike vectors X. If M contains a complete timelike line η then it is isometric to $R \times S$, where (S, h) is a complete Riemannian manifold [15].

2.2.3 Singularity theorems

Global hyperbolicity, geodesic completeness, limit curve theorem, singularity theorems are some of the essential ingredients essaying vital roles in the study of spacetime singularities. Penrose and S. Hawking derived singularity theorems, also known as Penrose-Hawking singularity theorems which provide sufficient conditions for the Lorentz manifold to be null and time geodesically incomplete. Let R_{ab} denote the components of *ricci* tensor. We state below four singularity theorems due to Penrose and Hawking [41, 34].

1. Let M be a globally hyperbolic spacetime with $R_{ab}X_aX_b \geq 0$ for all timelike vectors X, which will be the case if Einstein's equation is satisfied with the strong energy condition holding for matter. Suppose there exists a smooth (or, at least C^2) spacetike Cauchy surface Σ for which the trace of the extrinsic curvature (for the past directed normal geodesic congruence) satisfies $K \leq C < 0$

everywhere, where C is a constant. Then no past directed timelike curve from Σ can have length greater than $\frac{3}{|C|}$. In particular, all past directed timelike geodesics are incomplete.

2. Let M be a strongly causal spacetime with $R_{ab}X_aX_b \geq 0$ for all timelike vectors X. Suppose there exists a compact, edgeless, achronal, smooth spacelike hypersurface S such that for the past directed normal geodesic congruence from S we have $K < 0$ everywhere on S. Let C denote the maximum value of K, so $K \leq C < 0$ everywhere on S. Then at least one inextendible past directed timelike geodesic from S has length no greater than $\frac{3}{|C|}$.
3. Let M be a connected, globally hyperbolic spacetime with a non-compact Cauchy surface Σ. Suppose $R_{ab}X_aX_b \geq 0$ for all null vectors X as will be the case if (M, g) is a solution of Einstein's equation with matter satisfying the weak or strong energy condition. Suppose that M contains a trapped surface T. Let $\theta_0 < 0$ denote the maximum value of θ for both sets of orthogonal geodesics on T. Then at least one inextendible future directed orthogonal null geodesic from T has affine length no greater than $2/\theta_0$.
4. Suppose a spacetime M satisfies the following four conditions. (i) $R_{ab}X_aX_b \geq 0$ for all timelike and null vectors X, as will be the case if Einstein's equation is satisfied with the strong energy condition holding for matter, (ii) The timelike and null generic conditions are satisfied, (iii) No closed timelike curve exists, and (iv) At least one of the following three properties holds: (a) (M, g) possesses a compact achronal set without edge, (b) (M, g) possesses a trapped surface, (c) there exists a point $p \in M$ such that the expansion of the future (or past) directed null geodesics emanating from p becomes negative along each geodesic in this congruence. Then M must contain at least one incomplete timelike or null geodesic.

3 Survey of Relevant Literature

In this section, the study related to the proposed direction of work, as found in literature has been presented.

Over the past few decades, there have been various attempts at defining the physically relevant topologies on Lorentz manifolds. The most natural topology on Lorentz manifolds is the manifold topology. But this topology is inappropriate as: (i) it does not give the distinction between the spatial

and temporal directions, (ii) its homeomorphism group is very vast, (iii) the set of continuous paths contains many unphysical paths that allow switching between temporal and spatial direction.

Physically relevant non-Euclidean topologies on Minkowski space were first defined by Zeeman in 1967, namely *t*-topology, *s*-topology, *f*-topology, the space topology, the time topology and the fine topology. The fine topology is the finest topology that induces the 3-dimensional Euclidean topology on every spacelike hyperplane and 1-dimensional Euclidean topology on time axis. It enjoys properties like connectedness and locally connectedness. However, it is not locally compact. Its group of homeomorphisms is equal to the Lorentz group. Nanda proved this conjecture for space topology [38], t-topology [39] and s-topology [39]. He [38] also found that M is not normal, not locally compact and not second countable in the context of both space topology and s topology. Further, Agrawal et al. (2012) [10] obtained a characterization of the compact sets of *n*-dimensional Minkowski space with the *s*-topology and proved it to be not simply connectedness for 2-dimensional case using the notion of Zeno sequences which was initially introduced by Zeeman for the fine topology [4]. Also, *n*-dimensional Minkowski space with the *s*-topology was obtained to be a separable, first countable, path-connected, not regular, not metrizable, non-second countable, not compact and not Lindel*ö*f space. Continuing the topological study of *n*-dimensional Minkowski space with the *t*-topology, Agrawal et al. [9] characterized its compact sets and proved it to be a Hausdorff, path connected, separable, first countable but not second countable, not normal, not compact, not locally compact, not paracompact, not metrizable and not locally *n*-Euclidean space. Afterwards, Agrawal et al. (2015) obtained that invariance of domain theorem does not hold in Minkowski space with the *t*-topology unlike the Euclidean case. But its consequence, i.e., no nonempty open subset of the *n*-dimensional Minkowski space with the *t*-topology is homeomorphic to an open subset of the *m*-dimensional Minkowski space with the *t*-topology for $n \neq m$. The technique to prove these results is entirely different from that of Euclidean case [12]. Quite recently, S. Erosy [37] studied the interior and closure of some important subsets of Minkowski space with respect to the *s*-topology and the Euclidean topology. Continuing the topological study of Minkowski space, Agrawal et al. [8] studied the Minkowski space with the *f*-topology and obtained the characterization of its compact sets using the notion of Zeno sequences. It was obtained to be Hausdorff, path connected, separable, first

countable, non-regular, non-second countable, non-paracompact, non-Lindel*ö*f, non-normal, non-locally compact and non-metrizable space. For the two-dimensional case, it was obtained to be a non-simply connected space.

G*ö*bel, in 1976, defined general relativistic analogue of the fine topology (which he calls Zeeman topology) and obtained its group of homeomorphisms to be the same as the group of homothetic transformations on spacetime and found its world lines to be the broken geodesics [23]. In the same year, a seminar of Rudiger-G*ö*bel at Cambridge stimulated Hawking, King and McCarthy to introduce a physically relevant non-manifold path topology on strongly causal spacetime. It is found to be more appealing and manageable topology than Zeeman topology. It has been found that it is first countable, separable, Hausdorff, path connected, locally path connected but not regular, not normal, not locally compact and not paracompact. Also, it determines the causal, differential and conformal structure of spacetime. Malament proved that this result is true for every spacetime [1]. Naber found that the path topology on the four-dimensional Minkowski space, a strongly causal spacetime, is same as the Zeeman's *t*-topology [18]. Low proved that Minkowski space with the *t*-topology is not simply connected and generalized this result to Lorentz manifolds with the path topology [30].

G*ö*bel defined several non-manifold topologies on Lorentz manifolds and remarked that all these topologies are Hausdorff, locally and globally path connected and none of these topologies is regular, normal, locally compact and paracompact without given proofs [27]. One of these topologies is recognized as generalization of time topology on Minkowski space. He obtained Zeeman's conjecture in the context of the time topology on Minkowski space as a consequence of the result obtained for its generalized case on Lorentz manifolds. An alternate proof is obtained by Agrawal et al. in [11] using a simpler approach similar to one used in [18] for the computation of homeomorphism group of Minkowski space with the path topology. Fullwood worked out that the need of the manifold topology to define continuous timelike curves (used to define the path topology) restricts the class of spacetime on which the path topology can be defined [2]. He suggested a new topology on spacetime by combining the idea of Hawking's path topology and Alexandrov topology, which is completely defined in terms of causal structure. He also found that homeomorphism group of this new topology on spacetime having chronology condition is the group of chronological and anti-chronological

bijections and proved that this new topology determines the causal, differential and conformal structure for spacetime having distinguishing condition.

Penrose and Kronheimer (1967) introduced formally a causal space which is a set equipped with three relations, viz. causal, chronology and horismos satisfying certain axioms and suggested a way to attach causal boundary to spacetime which can be regarded as a causal space [41]. Ignatios et al. analysed the causal structure of the ambient boundary, the conformal infinity of the ambient (Poincar´e) metric. Using topological tools, they showed that the only causal relation compatible with the global topology of the boundary spacetime is the horismos order [19]. Later Minguzzi et al. provided a comprehensive study of the features causal spacetime satisfying various causality conditions thereby establishing a causal hierarchy of causal spacetimes [7]. Strong causality and global hyperbolicity are significant causality conditions defined for a causal space. Global hyperbolicity, limit curve theorem and upper semi continuous length functional are some of the few significant concepts which are studied in the context of Spacetime singularities [41]. Global hyperbolicity, the strongest causality condition can be defined both as the Lorentzian analogue of geodesic completeness in a Riemannian manifold and in terms of naked TIP (terminal indecomposable past) [16, 32].

R. Geroch first provided a splitting theorem for globally hyperbolic Lorentz manifold containing a compact Cauchy surface [6]. AN Bernal later obtained that a globally hyperbolic Lorentz manifold is diffeomorphic to $R \times S$, S being Cauchy hypersurface [6]. Cheeger and Gromoll proved that under certain conditions Riemannian manifold splits into the product $R \times V$, where (V, h) is a complete Riemannian manifold [17]. This motivated Yau to consider Lorentzian analogue of the Cheeger-Gromoll splitting theorem of Riemannian manifold [17]. Later on, Galloway, Beem, Eschenburg and Newman derived results in order to finally arrive at the result proposed by Yau [14, 15, 16]. Galloway had further obtained that global hyperbolicity and geodesic completeness in the Lorentz manifold are related with each other under suitable curvature conditions satisfied by the Lorentz manifold [16]. Low (in 1988) proved that a Lorentz manifold equipped with the manifold topology is globally hyperbolic if the space of causal curves having the topology induced from the manifold topology is Hausdorff [31] and at the same time established a connection between strong causality of a Lorentz manifold equipped with the manifold topology and the topological structure of null geodesics defined

on the manifold [31]. Low has further obtained the characterisation of globally hyperbolic Lorentz manifold equipped with the path topology in 2016 in terms of Hausdorffness of the space of causal curves [32].

Hawking and Ellis defined the first version of limit curve theorem which had some weak points; (i) It used a weak version of "limit curve", (ii) The convergence obtained does not allow to apply results on the upper semi-continuity of the length functional unless strong causality is assumed, (iii) It did not include the case of curves with both end points, not it included the case in which the limit event is unique [5]. Later, Beem et al. provided a stronger version of the limit curve theorem which proved an almost equivalence between C^0 convergence in a strongly causal space and limit curve convergence. However, it could not include the case when the sequence is made of curves with both end points. Therefore, Minguzzi had formulated a version of limit curve theorem which in a way was sufficiently general to serve as a solid reference for future applications [5].

4 Motivation

A smooth four-dimensional Lorentz manifold provides a mathematical framework for a study of general theory of relativity. Four-dimensional Minkowski space, a mathematical model for special theory of relativity, is the simplest example of a Lorentz manifold. Physically relevant non-Euclidean topologies, viz. fine topology, *t*-topology, *s*-topology, space topology and time topology defined on the Minkowski space were introduced by Zeeman. Study of topological properties of Minkowski space along with its various other aspects with respect to the above-mentioned topologies has gained momentum in recent years [9, 10, 11, 12]. Topologies on the Minkowski space were generalized on smooth Lorentz manifolds and were found to be consistent with its causal and conformal structure [35, 36]. For example, path topology [42] is the generalization of the t-topology and the geodesic topology is the generalization of the time topology. The proofs for properties of Minkowski space when generalized to Lorentz manifold with respect to path and geodesic topologies are found to be significantly difficult.

Singularity is the region of spacetime where laws of Physics break down [41]. The presence of singularity in spacetime is mathematically modelled by means of incompleteness, viz. geodesic-incompleteness, *b*-incompleteness and imprisoned incompleteness [41]. Limit curve theorem, global

hyperbolicity and upper semi-continuous length function are some of the essential ingredients defined for the Lorentz manifold that form the basis of most of the Penrose-Hawking singularity theorems [29]. Geodesic completeness and global hyperbolicity have been related to each other under some suitable conditions imposed on Lorentz manifold [16]. Globally hyperbolic Lorentz manifolds have been topologically characterised [34] and the result was improvised later [6]. Similarly, timelike geodesically complete Lorentz manifold satisfying certain conditions have been characterised [17]. Penrose and S. Hawking derived sufficient conditions for null and timelike geodesically incomplete spacetime. Geroch defined a topology on the collection of causal curves in a Lorentz manifold which was used by Penrose to prove the existence of a maximum length geodesic without conjugate points [41]. Topological study done by Low on the collection of null geodesics and causal curves defined on the Lorentz manifold has been proved to be a novel approach towards identifying the causality conditions, viz. global hyperbolicity and strong causality of the Lorentz manifold [31, 32]. In other words, causality conditions are reflected in the topological structure of the space of causal curves. Limit curve theorem plays a significant role in proving the upper semi-continuity of the length functional in a strongly causal spacetime. Moreover, different versions of this theorem have been formulated [28, 29]. In all this study, the topology under consideration is the manifold topology. Low studied a version of the limit curve theorem defined by Beem et al. for the Lorentz manifold equipped with the path topology [32] which was later studied by Papadopoulos for a Lorentz manifold equipped with other physically relevant non-manifold topologies [22]. Topological study of the space of causal curves has not been done so far.

The study proposed to be undertaken is aimed at carrying forward the study of the space of causal curves and various ingredients of singularity theorems in the context of the path topology.

5 Proposed Direction of Work

Geroch defined a topology known as *Geroch* topology, on the collection of causal curves in a Lorentz manifold endowed with the manifold topology [29]. Penrose used it to study the upper semi-continuity of the length function in a strongly causal Lorentz manifold [29]. Penrose-Hawking singularity theorems are concerned with deriving sufficient conditions for spacetime to be timelike and null geodesically incomplete in the context of the manifold topology and its essential ingredients are limit

curve theorem, global hyperbolicity, upper semi-continuity of length functional and Cauchy hypersurface. A version of the limit curve theorem, has been studied for the Lorentz manifold equipped with the path topology which is a physically significant topology on a Lorentz manifold introduced by Hawking et al. [42] strictly finer than the manifold topology. Also, Low has defined a topology which is different from the Geroch topology on the collection of causal curves in a strongly causal Lorentz manifold equipped with the path topology and studied global hyperbolicity of the Lorentz manifold in terms of Hausdorffness of the space of causal curves [31, 32].

The present proposal is focused on the study of Penrose-Hawking singularity theorems in the context of the path topology. The work is proposed to be carried out in the direction of the study of the following:

- Topological properties of the space of causal curves: separation and countability axioms, compact sets etc. for Minkowski space equipped with the *t*-topology and generalize this study to Lorentz manifold equipped with the path topology.
- Singularity theorems and related concepts for Lorentz manifolds in the context of Minkowski space equipped with the *t*-topology.
- Topological characterization of globally hyperbolic Lorentz manifold equipped with the path topology.
- Various versions of limit curve theorem in the context of the path topology.
- Penrose-Hawking singularity theorems for the Lorentz manifold equipped with the path topology.

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